Central sequence C^*-algebras and absorption of the Jiang-Su algebra

(Joint work with Eberhard Kirchberg)

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Outline

Let A be a unital C^*-algebra, and let ω be a free (ultra) filter on $\mathbb{N}$. Consider the central sequence C^*-algebra $A_\omega \cap A'$, where

$$A_\omega = \ell^\infty(A)/c_\omega(A), \quad c_\omega(A) = \{(x_n) \in \ell^\infty(A) \mid \lim_\omega \|x_n\| = 0\}.$$

What do we know about central sequence C^*-algebra $A_\omega \cap A'$?

Theorem (Kirchberg, 1994)

If A is a unital Kirchberg algebra (i.e., A is unital simple purely infinite separable and nuclear) and if ω is a free ultrafilter on $\mathbb{N}$, then $A_\omega \cap A'$ is simple and purely infinite.
In particular, $\mathcal{O}_\infty \hookrightarrow A_\omega \cap A'$, which entails that $A \cong A \otimes \mathcal{O}_\infty$.

Fact: $A \cong A \otimes \mathcal{Z} \iff$
$\exists$ unital *-homomorphism $\mathcal{Z} \to A_\omega \cap A'$ for some free filter ω.

Example

If A is unital and approximately divisible, then $\bigotimes_{k=1}^\infty (M_2 \oplus M_3)$ maps unitally into $A_\omega \cap A'$. Hence $\mathcal{Z} \hookrightarrow A_\omega \cap A'$, so $A \cong A \otimes \mathcal{Z}$.

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Fact: If M is a II_1 von Neumann factor and if ω is a free ultrafilter, then $M^\omega \cap M'$ is either a II_1 von Neumann algebra or it is abelian.

If the former holds, then M is said to be a *McDuff factor*, and in this case $\mathcal{R} \hookrightarrow M^\omega \cap M'$ which entails that $M \cong M \bar{\otimes} \mathcal{R}$.

Theorem (Strengthened version of a theorem of Sato)

Let A be a unital separable C^*-algebra with a faithful trace τ. Let $M = \pi_\tau(A)''$, and let ω be a free ultrafilter on $\mathbb{N}$. Then the canonical map

$$A_\omega \cap A' \to M^\omega \cap M'$$

is surjective.

- In particular, if A is non-elementary, unital, simple, nuclear and stably finite, then a quotient of $A_\omega \cap A'$ contains a subalgebra isomorphic to $\mathcal{R}$.

- Sato proved the theorem above under the additional assumptions that A is simple and nuclear.

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Idea of proof: The inclusion $A \to M$ induces a *-homomorphism $\Phi\colon A_\omega \to M^\omega$ which is *surjective* (by Kaplanski's density theorem).

Let $\pi_\omega\colon \ell^\infty(A) \to A_\omega$ be the quotient mapping and put $\widetilde{\Phi} = \Phi \circ \pi_\omega\colon \ell^\infty(A) \to M^\omega$.

Enough to show that if $b = (b_1, b_2, \dots) \in \ell^\infty(A)$ is such that $\widetilde{\Phi}(b) \in M^\omega \cap M'$, then $\exists c = (c_1, c_2, \dots) \in \ell^\infty(A)$ st $\widetilde{\Phi}(c) = \widetilde{\Phi}(b)$ and $\pi_\omega(c) \in A_\omega \cap A'$.

Put $D = C^*(A, b) \subseteq \ell^\infty(A)$ and put $J = \mathrm{Ker}(\widetilde{\Phi}|_D)$. Let $(e^{(k)}) \subseteq J$ be an asymptocially central approximate unit for J. Note that $ba - ab \in J$ for all $a \in A$. Hence, for all $a \in A$:

$$\begin{aligned} 0 &= \lim_{k\to\infty} \|(1 - e^{(k)})(ba - ab)(1 - e^{(k)})\| \\ &= \lim_{k\to\infty} \|(1 - e^{(k)})b(1 - e^{(k)})a - a(1 - e^{(k)})b(1 - e^{(k)})\|. \end{aligned}$$

We can therefore take $c_n = (1 - e_n^{(k_n)})b_n(1 - e_n^{(k_n)})$ for a suitable sequence (k_n).

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Example

There exist non-elementary, unital, simple, separable, nuclear (stably finite) C^*-algebras A that do not absorb the Jiang-Su algebra. E.g.:

- Villadsen's examples of simple AH-algebras with strongly perforated K_0-groups or with stable rank > 1.
- The example of a simple unital nuclear separable C^*-algebra with a finite and an infinite projection, [R], (which also provided a counterexample to the Elliott conjecture).
- Toms' refined counterexamples to the Elliott conjecture (which are AH-algebras).
- Many others!

For any of the C^*-algebras mentioned above, $\mathcal{Z}$ does not embed unitally into $A_\omega \cap A'$. For the stably finite ones, we still have a surjection $A_\omega \cap A' \to \mathcal{R}^\omega \cap \mathcal{R}'$, so $A_\omega \cap A'$ is not small (or abelian).

- When $\exists$ unital *-homomorphism $\mathcal{Z} \to A_\omega \cap A'$?
  - Not if $A_\omega \cap A'$ has a character.
  - Not if $A \not\cong A \otimes \mathcal{Z}$.
  - Perhaps if and only if $A_\omega \cap A'$ has no characters.

Proposition (Kirchberg: Abel Proceedings)

Let A and D be unital separable C^*-algebras, and let ω be a free filter on $\mathbb{N}$. If there is a unital *-hom $D \to A_\omega \cap A'$, then there is a unital *-hom

$$\bigotimes_{n=1}^{\infty} D \to A_\omega \cap A'$$

(where $\otimes = \otimes_{\max}$).

Corollary

If A is separable and $A_\omega \cap A'$ has no character, then $\exists$ unital separable C^*-algebra D with no characters and a unital *-homomorphism $\bigotimes_{n=1}^{\infty} D \to A_\omega \cap A'$.

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Theorem (Dadarlat–Toms)

Let D be a unital C^*-algebra. If $\bigotimes_{k=1}^{\infty} D$ contains a unital subhomogeneous C^*-algebra without characters, then $\mathcal{Z} \hookrightarrow \bigotimes_{k=1}^{\infty} D$.

- Consider the "dimension-drop" C^*-algebra:

$$I(2,3) := \{f : [0,1] \to M_2 \otimes M_3 \mid f(0) \in M_2 \otimes \mathbb{C},\ f(1) \in \mathbb{C} \otimes M_3\}.$$

Corollary

Let A be a separable unital C^*-algebra. TFAE:

- $A \cong A \otimes \mathcal{Z}$,
- $\exists$ unital $*$-homomorphism $\mathcal{Z} \to A_\omega \cap A'$,
- $A_\omega \cap A'$ contains unital subhomogeneous C^*-algebra without character,
- $\exists$ unital $*$-homomorphism $I(2,3) \to A_\omega \cap A'$.

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Corollary

Let A be a separable unital C^*-algebra. TFAE:

- $A \cong A \otimes \mathcal{Z}$,
- $\exists$ *unital* **-homomorphism* $\mathcal{Z} \to A_\omega \cap A'$,
- $A_\omega \cap A'$ *contains unital subhomogeneous* C^**-algebra without character,*
- $\exists$ *unital* **-homomorphism* $I(2,3) \to A_\omega \cap A'$.

Proposition (Perera–R, 2004)

Let A be a unital C^*-algebra of real rank zero. TFAE:

1. *A has no characters,*
2. *$\exists$ unital *-homomorphism $M_2 \oplus M_3 \to A$,*
3. *$\exists$ unital *-homomorphism $I(2,3) \to A$,*

- For *all* unital C^*-algebras: (2) $\Rightarrow$ (3) $\Rightarrow$ (1).
- (1) and (2) are not equivalent for non-real rank zero C^*-algebras, e.g., $A = C^*_r(\mathbb{F}_2)$.
- (1) and (3) are not equivalent in the non-real rank zero case. There are even simple infinite dimensional unital C^*-algebras that fail (3).
- It is not known if (1) and (3) are equivalent for unital C^*-algebras of the form $A = \bigotimes_{n=1}^{\infty} D$.

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Proposition: (R–Winter) $\exists$ unital *-hom $I(2,3) \to A_\omega \cap A'$ (and hence $A \cong A \otimes \mathcal{Z}$) if $\exists a, b \in A_\omega \cap A'$ positive contractions st

$$a \sim b, \qquad a \perp b, \qquad 1 - a - b \precsim (a - \varepsilon)_+,$$

i.e., if there exists *-hom $CM_2 \to A_\omega \cap A'$ with "large image".

Question

Let be A a unital separable C^*-algebra.

$$A_\omega \cap A' \quad \textit{has no characters} \overset{??}{\Longleftrightarrow} \exists \textit{ unital } {}^*\textit{-hom } \mathcal{Z} \to A_\omega \cap A'$$

$$\Longleftrightarrow A \cong A \otimes \mathcal{Z}$$

Question (Dadarlat–Toms)

Does $\mathcal{Z}$ embed unitally into $\bigotimes_{n=1}^{\infty} D$ whenever D is a unital C^*-algebra without characters?

- The two questions above are equivalent!

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Definition

A unital C^*-algebra is said to have the *splitting property* if there are positive full elements $a, b \in A$ with $a \perp b$.

Note: A has the splitting property $\implies$ A has no characters.

The opposite implication is false in general, but it may be true if $A = \bigotimes_{n=1}^{\infty} D$ for some unital D.

Lemma

If $A_\omega \cap A'$ has the splitting property, then there is a full *-homomorphism $CM_2 \to A_\omega \cap A'$.

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Proposition

Let A be a unital separable C^*-algebra and let ω be a free ultrafilter on $\mathbb{N}$.

(a) *If $A_\omega \cap A'$ has no character, then A has the strong Corona Factorization Property.*

(b) *If $A_\omega \cap A'$ has the splitting property, then $\exists N_k \in \mathbb{N}$ st*

1. $\forall k \geq 2\ \forall y \in \mathrm{Cu}(A)\ \exists x \in \mathrm{Cu}(A) : kx \leq y \leq N_k x$.
2. *Let $x, y \in \mathrm{Cu}(A)$. If $N_k x \leq ky$ for some $k \geq 1$, then $x \leq y$.*

(c) *If $A_\omega \cap A'$ has the splitting property and A is simple, then A is either stably finite or purely infinite.*

Corollary

There exist non-elementary, unital, simple, separable, nuclear C^*-algebras A st $A_\omega \cap A'$ has a character (and, at the same time, a sub-quotient $\cong \mathcal{R}$).

Outline

The Elliott classification conjecture for simple, separable, nuclear C^*-algebras does not hold without further assumptions. Good candidates for an additional assumption are the following:

Conjecture (Toms–Winter)

The following are equivalent for every separable simple nuclear (unital) C^*-algebra:

(a) *A has finite nuclear dimension.*

(b) *A has strict comparison of positive elements (or, equivalently, $\mathrm{Cu}(A)$ is almost unperforated).*

(c) $A \cong A \otimes \mathcal{Z}$.

- Winter proved (a) $\Rightarrow$ (c).
- (c) $\Rightarrow$ (b) holds for all C^*-algebras A, [R].
- (c) $\Rightarrow$ (a): listen to Sato's talk!

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In 2011, the following remarkable result was proved:

Theorem (Matui–Sato)

Let A be a unital, separable, simple, non-elementary, stably finite, nuclear C^*-algebra, and suppose that $\partial_e T(A)$ is finite. Then the following are equivalent:

(c) $A \cong A \otimes \mathcal{Z}$,

(b) *A has strict comparison of positive elements,*

(d) *Every cp map $A \to A$ can be excised in small central sequences,*

(e) *A has property (SI).*

We get back to the properties mentioned in (d) and (e).

Note that if A is not stably finite, then $T(A) = \emptyset$ and (b) implies that A is purely infinite. Hence A is a Kirchberg algebra and $A \cong A \otimes \mathcal{O}_\infty \cong A \otimes \mathcal{Z}$.

It would be desirable to remove the condition that $\partial_e T(A)$ is finite!

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It would be desirable to remove the condition that $\partial_e T(A)$ is finite!

A unital C^*-algebra with $T(A) \neq \emptyset$. Define

$$\|a\|_{2,\tau} = \tau(a^*a)^{1/2}, \qquad \|a\|_2 = \sup_{\tau \in T(A)} \|a\|_{2,\tau}, \quad a \in A.$$

Define $\| \cdot \|_2$ on A_ω by

$$\|\pi_\omega(a_1, a_2, a_3, \dots)\|_2 = \lim_\omega \|a_n\|_2,$$

where $\pi_\omega \colon \ell^\infty(A) \to A_\omega$ is the quotient map. Set

$$J_A = \{x \in A_\omega : \|a\|_2 = 0\} \ \lhd \ A_\omega.$$

Definition (Matui–Sato, reformulated)

A unital simple C^*-algebra A is said to have *property (SI)* if for all positive contractions $e, f \in A_\omega \cap A'$ such that

$$e \in J_A, \qquad \sup_k \|1 - f^k\|_2 < 1,$$

there is $s \in A_\omega \cap A'$ with $fs = s$ and $s^*s = e$.

Proposition

Let A be a separable, simple, unital, stably finite C^*-algebra with property (SI). TFAE:

1. $A \cong A \otimes \mathcal{Z}$,
2. $\exists$ *unital* **-homomorphism* $\mathcal{R} \to (A_\omega \cap A')/J_A$.
3. $\exists$ *unital* **-homomorphism* $M_2 \to (A_\omega \cap A')/J_A$.
4. $\exists$ *unital* **-homomorphism* $I(2,3) \to A_\omega \cap A'$.

Proposition

If A is a non-elementary, unital, simple, separable, stably finite C^*-algebra st

1. $\pi_\tau(A)''$ *is McDuff factor for all* $\tau \in \partial_e T(A)$.
2. $\partial_e T(A)$ *is weak* * *closed in* $T(A)$ *(i.e., $T(A)$ is a Bauer simplex).*
3. $\partial_e T(A)$ *has finite covering dimension.*

Then there is a unital **-homomorphism* $M_2 \to (A_\omega \cap A')/J_A$.

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Results similar to the ones above and below have been obtained independently by A. Toms, S. White and W. Winter and by Y. Sato.

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Then $A \cong A \otimes \mathcal{Z}$.

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Definition (Matui–Sato, reformulated)

A cp map $\varphi\colon A \to A \subseteq A_\omega$ can be *excised in small central sequences* if for all positive contractions $e, f \in A_\omega \cap A'$ with

$$e \in J_A, \qquad \sup_k \|1 - f^k\|_2 < 1,$$

there exists $s \in A_\omega$ st

$$fs = s, \qquad s^* as = \varphi(a)e, \qquad a \in A.$$

Proposition (Matui–Sato)

Let A be a unital simple C^*-algebra.

1. *If $\mathrm{id}_A\colon A \to A$ can be excised in small central sequences, then A has property (SI).*
2. *If A is simple, separable, unital and nuclear, and if A has strict comparison, then id_A can be excised in small central sequences.*

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Definition

Let A be a unital, simple, stably finite C^*-algebra. Then A has *local weak comparison* if there exists a constant $\gamma = \gamma(A)$ st for all positive element $a, b \in A$:

$$\gamma \cdot \sup_{\tau \in QT(A)} d_\tau(a) < \inf_{\tau \in QT(A)} d_\tau(b) \implies a \precsim b.$$

- A has strict comparison $\iff \mathrm{Cu}(A)$ is almost unperforated $\implies \mathrm{Cu}(A)$ has strong tracial m-comparison for some $m < \infty$ (in the sense of Winter) $\implies A$ has local weak comparison.
- A is $(m, \bar{m})$-pure (in the sense of Winter) for some $m, \bar{m} \in \mathbb{N}$ $\implies A$ has local weak comparison.

Theorem (Winter)

If A is simple, separable, unital with locally finite nuclear dimension, and if A is $(m, \bar{m})$-pure for some $m, \bar{m} \in \mathbb{N}$, then $A \cong A \otimes \mathcal{Z}$.

Definition

Let A be a unital, simple, stably finite C^*-algebra. Then A has *local weak comparison* if there exists a constant $\gamma = \gamma(A)$ st for all positive element $a, b \in A$:

$$\gamma \cdot \sup_{\tau \in QT(A)} d_\tau(a) < \inf_{\tau \in QT(A)} d_\tau(b) \implies a \precsim b.$$

- A has strict comparison $\iff$ $\mathrm{Cu}(A)$ is almost unperforated $\implies$ $\mathrm{Cu}(A)$ has strong tracial m-comparison for some $m < \infty$ (in the sense of Winter) $\implies$ A has local weak comparison.
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Proposition

Let A be a unital, simple, stably finite C^*-algebra.

1. *If A has local weak comparison, then every nuclear cp $\varphi\colon A \to A$ can be excised in small central sequences.*
2. *If A is nuclear and has local weak comparison, then A has property (SI).*

Corollary

Let A be a non-elementary, stably finite, simple, separable, unital and nuclear C^*-algebra. Suppose that $\partial_e T(A)$ is closed in $T(A)$ and that $\partial_e T(A)$ has finite covering dimension. Then the following are equivalent:

(b') *A has local weak comparison,*

(b) *A has strict comparison of positive elements,*

(c) $A \cong A \otimes \mathcal{Z}$.

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Question

Are (b'), (b) and (c) above equivalent for all non-elementary, simple, separable, unital and nuclear C^*-algebra?

Are (b) and (b') above equivalent for all separable, unital and nuclear C^*-algebra?

Outline

A bit about the proof of (1) of:

Proposition

Let A be a unital, simple, stably finite C^*-algebra.

1. *If A has local weak comparison, then every nuclear cp $\varphi\colon A \to A$ can be excised in small central sequences.*
2. (Matui–Sato) *If* id_A *can be excised in small central sequences, then A has property (SI).*

Definition (Matui–Sato, reformulated)

A cp map $\varphi\colon A \to A \subseteq A_\omega$ can be *excised in small central sequences* if for all positive contractions $e, f \in A_\omega \cap A'$ with

$$e \in J_A, \qquad \sup_k \|1 - f^k\|_2 < 1,$$

there exists $s \in A_\omega$ st

$$fs = s, \qquad s^*as = \varphi(a)e, \qquad a \in A.$$

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Definition

Let $A \subseteq B$. A cp map $\varphi\colon A \to B$ is *one-step-elementary* if there exist a pure state λ and $c_1, \ldots, c_n, d_1, \ldots, d_n \in B$ st

$$\varphi(a) = \sum_{j,k=1}^{n} \lambda(d_j^* a d_k) c_j^* c_k.$$

Lemma (cf. Matui–Sato, Prop. 2.2)

If A is a unital simple separable C^*-algebra with $QT(A) = T(A)$ and with local weak comparison, then every one-step-elementary cp map $\varphi\colon A \to A$ can be excised in small central sequences.

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Proof of Lemma

Let

$$\varphi(a) = \sum_{j,k=1}^{n} \lambda(d_j^* \, a \, d_k) c_j^* c_k,$$

be a one-step-elementary cp map. Take $h \in A^+$ with $\|h\| = 1$ that excises the state λ as follows:

$$\|h^{1/2}(x - \lambda(x) \cdot 1) h^{1/2}\| < \frac{1}{2}\Big(\sum_{j=1}^{n} \|c_j\|\Big)^{-2} \varepsilon$$

for all $x \in \{d_j^* a d_i \mid 1 \leq i, j \leq n,\ a \in F\}$.

Let $t \in A_\omega$ be such that $t^* h t = e$, $ft = t$, and $\|t\|^2 \leq 2$. Put

$$s = \sum_{i=1}^{n} d_i h^{1/2} t c_i.$$

Then $fs = s$ and

$$\|s^* a s - \varphi(a) e\| = \Big\| \sum_{j,k=1}^{n} c_j^* t^* h^{1/2} \big(d_j^* a d_k - \lambda(d_j^* a d_k)\big) h^{1/2} t c_k \Big\| < \varepsilon.$$

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To prove the claim we are after: *If A is unital, simple, stably finite and has local weak comparison, then every nuclear cp $\varphi \colon A \to A$ can be excised in small central sequences* we need now only prove that:

Proposition

Let A be a separable (simple) non-elementary C^*-algebra. Then every nuclear cp map $\varphi \colon A \to A$ is the point-norm limit of a sequence of one-step-elementary cp maps $A \to A$.

Definition

Let $A \subseteq B$. A cp map $\varphi \colon A \to B$ is *one-step-elementary* if there exist a pure state λ and $c_1, \ldots, c_n, d_1, \ldots, d_n \in B$ st

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Proposition

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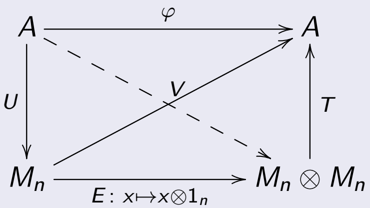
Proof.



- $T(y) = C^*yC$ for some $C \in M_{n^2,1}(A)$.
- $S := E \circ U$ can be approximated by maps $S'\colon A \to M_{n^2}$ of the form
$$S'(a) = \big[\lambda(d_j^* a d_k)\big]_{1\le i,j\le n^2}, \quad a \in A.$$

▶ $T \circ S'\colon A \to A$ is one-step-elementary, and $T \circ S' \approx \varphi$. □